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Unsteady Transonic Nozzle Flow with Heat Addition

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Transonic flows are very sensitive to heat release due to condensation, combustion, or electrical heating. The governing unsteady small disturbance equations for such flows are formulated in this paper. In particular, a slowly time-varying regime in which the characteristic disturbance time is much larger than a characteristic flow time is considered. In the case in which the unsteady heat release is spatially homogeneous, a similarity solution for flow in a converging-diverging nozzle has been found. The response of the flow to various forms of unsteady heat input has been computed, and the role of shock waves within the nozzle is considered.

Introduction

UNSTEADINESS or instability can be triggered in transonic flows by a relatively small release of heat due to such processes as condensation, combustion, or electrical heating. Thus, flow oscillations induced by the heat released during condensation of water vapor near the throat of converging-diverging nozzles have been observed by Schmidt,[1] Barschdorff,[2-4] and Wegener and Cagliostro.[5] Transonic condensation effects also play an important role in determining the operating limits of cryogenic high Reynolds number transonic tunnels.[6] Unsteady transonic flow with heating also arises near the throat of the nozzles employed in certain air blast circuit breakers.[7] In the present paper, the transonic small disturbance equations for such flows are formulated and an exact similarity solution of these equations for a particular form of heat addition is presented.

Flows with processes resulting in the evolution of heat have been the subject to extensive investigation as discussed in the review by Becker,[8] and the monograph by Zierep,[9] for example. The equations for steady transonic flow of reactive gases have been formulated and discussed by Napolitano[10] and Prud'Homme.[11] However, aside from the approximate analysis of Barschdorff and Filipov,[12] a transonic theory applicable to the type of unsteady flows just described does not appear to be available.

The present development starts with the basic equations for reactive or nonequilibrium flow from which the transonic small disturbance equations are then derived. The particular case in which the characteristic disturbance time $\tau_{ch} \gg \tau_f$, the characteristic flow time is considered since the phenomena described above lie in this "slowly time varying" regime. The resultant equations are identical to the unsteady transonic small disturbance equations derived by Adamson[13] except for an added reactive term. An insight into the character of such flows is provided by a self-similar solution of these equations for flow in a converging-diverging nozzle.

Basic Formulation

Inviscid but nonequilibrium flows will be considered, so that the equations for the conservation of mass, momentum, and energy are

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho \nabla \cdot v = 0 \tag{1}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} + \nabla p / \rho = 0 \tag{2}$$

$$\frac{\mathrm{D}h}{\mathrm{D}t} - \frac{1}{\rho}\frac{\mathrm{D}p}{\mathrm{d}t} = \frac{\dot{q}}{\rho} \tag{3}$$

Here $\dot{q}$ is the rate of external heating per unit volume due to such processes as Joule heating or radiation, and will, in general, be a function of both position and time.

In many important processes a single nonequilibrium variable, say ξ, is sufficient, and the present discussion will be restricted to this case. Thus, a caloric equation of state can be written as

$$h = h(p, \rho, \xi) \tag{4}$$

Combining Eqs. (1-4) and using the relation[14]

$$a_f^2 = \left(\frac{\partial p}{\partial \rho}\right)_{s,\xi} = -\left(\frac{\partial h}{\partial \rho}\right)_{p,\xi} \Big/ \left[\left(\frac{\partial h}{\partial p}\right)_{\rho,\xi} - \frac{1}{\rho}\right] \tag{5}$$

for the frozen speed of sound then leads to the relation

$$v \cdot \frac{\mathrm{D}v}{\mathrm{D}t} - a_f^2 \nabla \cdot v - \frac{1}{\rho}\frac{\partial p}{\partial t} + \frac{a_f^2}{p(\partial h/\partial \rho)_{p,\xi}} \times \left[\left(\frac{\partial h}{\partial \xi}\right)_{p,\rho}\frac{\mathrm{D}\xi}{\mathrm{D}t} - \frac{\dot{q}}{\rho}\right] = 0 \tag{6}$$

This equation, which forms the point of departure for the analysis that follows, reduces to the well-known gasdynamic equation in the case of steady nonreactive flow. In steady nonreactive flow with external heating, Eq. (6) is identical to that discussed by Zierep,[9] among others.

Equation (6) is completely general and can be used to treat flows with condensation, combustion, vibrational relaxation, or Joule heating. However, the definition of ξ, the thermodynamic derivatives, the form of $\dot{q}$, and the rate law governing $\mathrm{D}\xi/\mathrm{D}t$ will depend on the process under consideration. For example, in the case of a condensing vapor in an inert gas, such as water vapor in air, the appropriate choice for ξ is g, the mass fraction of condensate. From the thermodynamics of condensing flows,[15] it then follows that:

$$\frac{a_f^2}{\rho}\frac{\left(\frac{\partial h}{\partial g}\right)_{p,\rho}}{\left(\frac{\partial h}{\partial \rho}\right)_{p,g}} = \frac{\left[L - \frac{C_{p0}T}{(1-g)(\mu_v/\mu)}\right](1-g)\frac{R}{\mu}}{C_{p0} - (1-g)\frac{R}{\mu}} \tag{7}$$

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where R is the universal gas constant, μ the average molecular weight of the gaseous part of the mixture, L the latent heat of vaporization, C_{p0} the constant pressure specific heat of the initial air vapor mixture, while μ_v is the vapor molecular weight. On the other hand, for external heating of an inert gas $D\xi/Dt=0$, and

$$-\frac{a_f^2}{\rho(\partial h/\partial \rho)_{p,\xi}}\frac{\dot{q}}{\rho}=(\gamma-1)\frac{\dot{q}}{\rho} \tag{8}$$

where γ is the ratio of specific heats.

Equations (1-3) and (6) can be simplified in transonic flow because only small perturbations from the sonic velocity need to be considered. However, within the transonic regime of flow, certain restrictions are placed on the magnitude of the reactive and heating terms in Eq. (6) regardless of the process involved. To make further progress the equations given must now be specialized to transonic flow.

Transonic Small Disturbance Equations

The subsequent development parallels that of Adamson[13] in the nonreactive case. Small deviations from a uniform flow moving with velocity U in the x direction, and with density and temperature ρ_0 and T_0 are considered. The undisturbed flow is taken close to the sonic velocity so that $(U-a_f)/a_f \ll 1$, and the ordering of the dependent variables and stretching of the independent variables is taken to be the same as in the case of inert flow.[13] The present treatment is restricted to two-dimensional flow.

The velocity, density, temperature, and pressure are made dimensionless using $\bar{U}$, $\bar{\rho}_0$, $\bar{T}_0$ and $\bar{\rho}_0\bar{U}^2$ as reference quantities, and for the development to follow dimensional quantities will be denoted by overbars. Then appropriate expansions for u and v, the x and y velocity components and for the density ρ, pressure p, temperature T, and frozen speed of sound a_f are

$$\left.\begin{aligned} u&=1+E_1u^{(1)}+\dots; & v&=\delta E_1 v^{(1)}+\dots\\ \rho&=1+E_1\rho^{(1)}+\dots; & p&=p^{(0)}+E_1p^{(1)}+\dots\\ T&=1+E_1T^{(1)}+\dots; & a_f&=a_f^{(0)}+E_1a_f^{(1)}\end{aligned}\right\} \tag{9}$$

E_1 is a small parameter characterizing deviations from the undisturbed flow while, following the inert ordering, the parameter δ is related to E_1 by $\delta^2=E_1(\gamma+1)$. Stretched $\bar{x}$ and $\bar{y}$ coordinates are given by

$$x=\bar{x}/\delta\bar{\ell}; \quad y=\bar{y}/\bar{\ell} \tag{10}$$

This stretching accounts for the different rates of change in the x and y directions that characterize transonic flow. Since nozzle flows are of prime concern here, the characteristic length $\bar{\ell}$ is taken as the nozzle half-height. It will be seen later that Eq. (10) confines the analysis to a narrow region of the nozzle throat.

As already mentioned, the present analysis is restricted to the slowly time varying regime so that $\bar{\tau}_{ch} \gg \bar{\tau}_f = \bar{\ell}/\bar{U}$. Following Adamson[13] the dimensionless time

$$t=k\delta\bar{t}/(\bar{\ell}/\bar{U}) \tag{11}$$

is therefore introduced. Here k is a constant of 0(1). Equation (11) implies that the Strouhal number $Str=\bar{\tau}_f/\bar{\tau}_{ch}=k\delta \ll 1$.

At this point it is most convenient to use the conservation equations in their original form. Substituting the stretched variables and expansions in Eq. (1) and in the x momentum equation then yields the relations

$$u^{(1)}+\rho^{(1)}=0; \quad u^{(1)}+p^{(1)}=0 \tag{12}$$

provided the undisturbed flow is uniform and steady. Together with Eqs. (12) the y momentum equation then yields the result

$$\frac{\partial v^{(1)}}{\partial x}-\frac{\partial u^{(1)}}{\partial y}=0 \tag{13}$$

Hence the flow is irrotational to first order.

In order to deal with the energy equation, it is necessary to specify the forms of the caloric equation of state Eq. (4). In the present case it is reasonable to assume that the fluid is a perfect gas with constant specific heats. For a condensing vapor in an inert gas,[15] Eq. (4) then becomes

$$\bar{h}=\bar{C}_p\bar{T}-g\bar{L}=[\gamma p/(\gamma-1)\rho]-g\bar{L} \tag{14}$$

where g is now the nonequilibrium variable, and the second term on the right of this expression is dropped if only external heating is under consideration. Keeping only the largest terms, the energy equation now becomes

$$\frac{E_1}{\delta}\left[\frac{\partial T^{(1)}}{\partial x}-(\gamma-1)M_0^2\frac{\partial p^{(1)}}{\partial x}\right]=\frac{\bar{\tau}_f}{\bar{C}_p\bar{T}_0}\left[\frac{\dot{\bar{q}}}{\bar{\rho}}-\left(\frac{\partial\bar{h}}{\partial g}\right)_{\bar{T}}\frac{Dg}{D\bar{t}}\right] \tag{15}$$

where $M_0=\bar{U}_0/\bar{a}_{f0}$. The order of magnitude of the heating term on the right-hand side will be established by considering the gasdynamics Eq. (6).

Keeping only the largest terms, the gasdynamics Eq. (6) becomes

$$\begin{aligned}&\delta E_1\left(k\frac{\partial u^{(1)}}{\partial t}-k\frac{\partial p^{(1)}}{\partial t}-a_f^{(0)2}\frac{\partial v^{(1)}}{\partial y}\right)\\ &+\frac{E_1^2}{\delta}\left(\frac{1-a_f^{(0)2}}{E_1}+2u^{(1)}-2a_f^{(1)}a_f^{(0)}\right)\frac{\partial u^{(1)}}{\partial x}\\ &=-\frac{a_f^{(0)2}\bar{\tau}_f}{\bar{C}_p\bar{T}_0}\left[\frac{\dot{\bar{q}}}{\bar{\rho}_0}-\left(\frac{\partial\bar{h}}{\partial g}\right)_{p,\rho}\frac{Dg}{D\bar{t}}\right]\end{aligned} \tag{16}$$

The left-hand side of this equation is of $0(\delta E_1)$; hence, it follows that, to be consistent with the transonic approximation, the reactive and heating terms on the right can be at most of $0(\delta E_1)$. From Eq. (7) it follows that the right side of Eqs. (15) and (16) are of the same order. The right side of the energy Eq. (15) can, therefore, be dropped to first order, and then one integration yields the relation

$$T^{(1)}-(\gamma-1)M_0^2p^{(1)}=0 \tag{17}$$

The gasdynamics equation can now be simplified further using the results developed so far. From Eqs. (12) and (17) and the ideal gas equation, it follows that

$$p^{(1)}=-(1/M_0^2)u^{(1)} \tag{18}$$

In the case of an ideal gas substitution of the expansions Eq. (9) in Eq. (5) for $\bar{a}_f^2$, together with Eqs. (12) and (17), yields the following expression:

$$\begin{aligned}a_f^2&=(1/M_0^2)(1+E_1T^{(1)}+\dots)\\ &=(1/M_0^2)[1-E_1(\gamma-1)M_0^2u^{(1)}+\dots]\end{aligned} \tag{19}$$

Since the flow is irrotational to first order, it is possible to define the potential

$$\bar{\phi}_1=U\bar{\ell}[(\bar{x}/\bar{\ell})+E_1\delta\phi(x,y,t)+\dots] \tag{20}$$

so that

$$u^{(1)}=\phi_x; \quad v^{(1)}=\phi_y \tag{21}$$

Utilizing the result that $M_0^2 = 1 + 0(E_l)$, the gasdynamics equation can now be reduced to the potential equation

$$2k\phi_{xt} + (\chi_s + \phi_x)\phi_{xx} - \phi_{yy}$$

$$= -\frac{1}{(E_l\delta)}\frac{\bar{\tau}_f}{\bar{C}_p\bar{T}_0}\left[\frac{\dot{\bar{q}}}{\bar{\rho}_0} - \left(\frac{\partial\bar{h}}{\partial g}\right)_{p,\rho}\frac{Dg}{D\bar{t}}\right] \tag{22}$$

where $\chi_s = (M_0^2 - 1)/(\gamma+1)E_l$, and the subscripts denote differentiation.

The left side of Eq. (22) is identical to the unsteady transonic small disturbance equation for inert flow.[13] The heating term on the right requires further discussion and is considered below.

Since all transport effects are neglected here, the boundary conditions at a solid surface will be the same as in the inert case considered by Adamson.[13] Hence, with the wall coordinate y_w given by

$$y_w = y_i + (\gamma+1)E_l^2 F_w(x,t) \tag{23}$$

the wall boundary condition is

$$v_w^{(1)} = (\phi_y)_w = (\partial F_w/\partial x) \tag{24}$$

Here y_i is a constant.

Application to Nozzle Flow with Heating

As previously indicated, the application of greatest interest here is transonic nozzle flow with heating from condensation or an external source. The relation of the small disturbance Eq. (22) to such flows now requires further discussion. Distinguishing features of this equation are the heating term on the right which limits the processes that can be treated, and the restriction of the unsteady processes to the slowly time varying regime.

To begin, the small parameter E_l will be related to the geometry of the nozzle throat where the flow of interest occurs. The throat section of a nozzle with wall radius of curvature $\bar{R}$ is shown in Fig. 1. The ordinate of the wall contour is given by Eq. (23) from which it follows that

$$\frac{1}{\bar{R}} = \frac{d^2\bar{y}_w}{d\bar{x}^2} = (\gamma+1)E_l^2\frac{1}{\delta^2\bar{\ell}}\frac{\partial^2 F_w}{\partial x^2} = \frac{E_l}{\bar{\ell}}\frac{\partial^2 F_w}{\partial x^2} \tag{25}$$

since $\delta^2 = E_l(\gamma+1)$. Equation (25) is valid as long as $(d\bar{y}_w/d\bar{x})^2 \ll 1$. Since F_w, x, and t, are all 0(1), it follows that $(\partial^2 F_w/\partial x^2) \sim 0(1)$. Hence

$$E_l \sim 0(\bar{\ell}/\bar{R}) \tag{26}$$

and a relation between E_l and the nozzle geometry thus is established.

With this value of E_l it can be shown that physically observed flow oscillations in nozzles fall within the slowly time varying regime. Thus, Wegener and Cagliostro[5] found that the frequency of condensation induced oscillations could be correlated in terms of the dimensionless frequency f defined by

$$f = \bar{f}(2\bar{R}\bar{\ell})^{1/2}/\bar{a}^* \tag{27}$$

where $\bar{f}$ is the actual oscillation frequency and $\bar{a}^*$ the critical speed of sound. For the data reported,[5] $0.185 < f < 0.940$ or, in other words, $f \sim 0(1)$. Now, using $E_l \sim 0(\bar{\ell}/\bar{R})$, $\bar{\tau}_f = \bar{\ell}/\bar{a}^*$, and $\bar{\tau}_{ch} = (1/\bar{f})$, it follows from Eq. (27) that

$$\bar{\tau}_f/\bar{\tau}_{ch} \sim 0(E_l^{1/2} f) \sim 0(\delta) \tag{28}$$

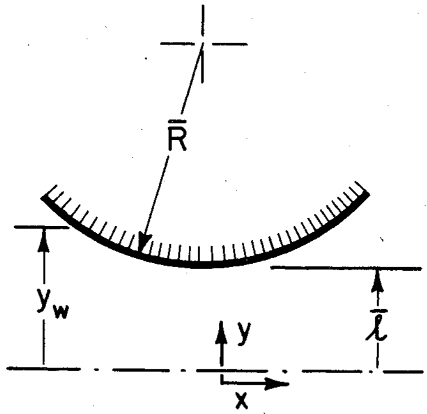

Fig. 1 Nozzle parameters.

so that the observed result that $f \sim 0(1)$ implies that the nozzle oscillations were indeed within the slowly time varying regime. The actual oscillation frequencies $\bar{f}$ observed in the slowly time varying regime are by no means small. In a nozzle with $\bar{\ell} = 0.65$ cm, $\bar{R} = 16$ cm, and with $\bar{a}^* \simeq 300$ m/s frequencies up to 6000 Hz were observed.[5] Other parameters reported by Wegener and Cagliostro were also found to be consistent[16] with the ordering used to develop Eq. (22).

The heating term on the right of Eq. (22) cannot exceed 0(1) within the order of transonic approximation considered here. For purely external heating it then follows that

$$\dot{\bar{q}}/\bar{\rho}_0\bar{C}_p\bar{T}_0 \sim 0(E_l\delta/\bar{\tau}_f) \sim 0(E_l/\bar{\tau}_{ch}) \tag{29}$$

If the external heating rate can be characterized by heat $\bar{q}_0$ added per unit volume of flow in a heating time $\bar{\tau}_h$, it follows from Eq. (29) that

$$\bar{q}_0/\bar{\rho}_0\bar{C}_p\bar{T}_0 \sim 0(E_l\bar{\tau}_h/\bar{\tau}_{ch}) \tag{30}$$

or, with $\bar{\tau}_h \sim \bar{\tau}_{ch} = (\bar{\tau}_f/\delta)$, the dimensionless heat addition $\bar{q}_0/\bar{\rho}_0\bar{C}_p\bar{T}_0$ can be at most of $0(E_l)$.

If heating is from condensation alone, the restriction becomes

$$\frac{1}{E_l\delta}\frac{\bar{\tau}_f}{\bar{C}_p\bar{T}_0}\left(\frac{\partial\bar{h}}{\partial g}\right)_{p,\rho}\frac{Dg}{D\bar{t}} \sim 0(1) \tag{31}$$

From the thermodynamics of vapor-inert mixtures, it can be shown that

$$\left(\frac{\partial h}{\partial g}\right)_{p,\rho} \sim 0(\bar{L}) \tag{32}$$

If ω_0 is the specific humidity prior to condensation and $\bar{\tau}_c$ is a characteristic condensation time, $Dg/D\bar{t} \sim 0(\omega_0/\bar{\tau}_c)$, then Eq. (31) implies the following relations between $\bar{\tau}_c$ and the characteristic disturbance time $\bar{\tau}_{ch}$.

$$\frac{\bar{\tau}_c}{\bar{\tau}_{ch}} \sim 0\left(\frac{\omega_0 D}{E_l}\right) \tag{33}$$

where $D = \bar{L}/\bar{C}_{p0}\bar{T}_0 \simeq (\gamma-1)\bar{L}/\bar{a}^{*2}$.

In Eq. (33) E_l and $\bar{\tau}_{ch}$ depend on the geometry of the nozzle and on $\bar{a}^*$, $\bar{L}$ is a physical property of the fluid and ω_0 an externally imposed experimental condition. The condensation time $\bar{\tau}_c$, on the other hand, depends on the physics of the condensation process and is essentially independent of the other parameters in Eq. (33). This equation, therefore, does not determine $\bar{\tau}_c$ in terms of ω_0, D, $\bar{\tau}_{ch}$, and E_l but simply indicates for what order of $\bar{\tau}_c$ the condition Eq. (31) is satisfied. The dimensions of the nozzles studied by Wegener

Table 1 Dimensions and characteristic times of nozzles used by Wegener and Cagliostro[5] and by Barschdorff[2,3]

$D \cong 11$; $\bar{a}^* \cong 300$ m/s; $\gamma = 1.4$							
$\bar{\ell}$, cm	$\bar{R}$, cm	$\bar{\ell}/\bar{R}$ (κ, E_I)	δ	$\bar{\tau}_f$, μs	$\bar{\tau}_{ch}$, μs	$\bar{\tau}_c$, μs $\omega_0 = 0.01$	$\bar{\tau}_c$, μs $\omega_0 = 0.02$
0.65	16	0.041	0.313	22	70	193	386
0.65	18	0.036	0.294	22	75	236	472
2.55	15	0.170	0.639	85	133	89	178
3.00	20	0.150	0.600	100	167	126	252
3.00	58	0.052	0.353	100	353	769	1538

and Cagliostro[5] and by Barschdorff as well as corresponding values of E_I, $\bar{\tau}_f$, $\bar{\tau}_{ch}$ and values of $\bar{\tau}_c$ computed using Eq. (33) are presented in Table 1. In these experiments $\bar{a}^* \simeq 300$ m/s, $D \simeq 11$, and $0.01 < \omega_0 < 0.02$. It can be seen that the values of $\bar{\tau}_c$ determined from Eq. (33) fall into the range $90 \leq \bar{\tau}_c \leq 1500$ μs. On the other hand, Wegener and Cagliostro indicate that the characteristic time for condensation lies in the range $10 < \bar{\tau}_c' < 100$ μs, where the physically observed condensation time is denoted by a prime to distinguish it from the value given by Eq. (33). In the upper range, $\bar{\tau}_c'$ satisfies condition Eq. (33) for many of the cases given in Table 1. However, when $\bar{\tau}_c' \simeq 10$ μs, $\bar{\tau}_c' \ll \bar{\tau}_c$ and Eq. (31) will be violated. The small disturbance equation in the form of Eq. (22) will no longer be applicable, and a different treatment probably involving narrow condensation fronts will be required. The presence of such fronts is, in fact, indicated in interferograms of nozzle flows with condensation.[2]

Similarity Solution—Formulation

Athough solutions of the small disturbance Eq. (22) for condensation induced oscillatory flows are not yet available, it is possible to develop an exact similarity solution of Eq. (22) for special forms of external heating, $\dot{q}$. This solution, which describes the type of flow generated by electrical or Joule heating, and provides valuable insight into the effects of heating upon transonic nozzle flow, is developed next.

At first, it is convenient to express Eq. (22) in terms of the velocity perturbation $u^{(1)}$, and to replace the external heating term by the heating function

$$Q(x,y,t) = (\bar{\tau}_f \dot{\bar{q}}) / (E_I \delta \bar{C}_p \bar{T}_0 \bar{\rho}_0) \tag{34}$$

Taking $M_0 = 1$ so that $\chi_s = 0$, and differentiating with respect to x

$$2k\phi_{xxt} + (\phi_x \phi_{xx})_x - \phi_{yyx} + Q_x = 0 \tag{35a}$$

In terms of $u^{(1)} = \phi_x$, Eq. (35a) can be reduced to

$$2ku^{(1)}{}_{xt} + (u^{(1)} u^{(1)}{}_x)_x - u^{(1)}_{yy} + Q_x = 0 \tag{35b}$$

Introducing the transformation

$$S = x + by^2 + \beta(t) \qquad u^{(1)} = Z(S) + 4b^2y^2 - 2k\beta'(t) \tag{36}$$

first used by Adamson[13] reduces Eq. (35b) to the following equation for the function $Z(S)$:

$$(ZZ')' - 2bZ' - 8b^2 + Q_x = 0 \tag{37}$$

When Q is a function of time t alone, $Q_x = 0$ and Eq. (37) is reduced to the ordinary differential equation

$$(ZZ')' - 2bZ' - 8b^2 = 0 \tag{38}$$

The function $\beta(t)$ is an arbitrary function of time, and b is a constant whose significance will become evident later. The flow described by Eq. (36) is symmetrical with respect to the x axis; thus, any two streamlines mirrored in the x axis can be taken as the walls of a nozzle. The function $Z(S)$ is related to the velocity perturbation $u^{(1)}$ on the axis $y = 0$, and $Z(S) = u^{(1)}(x,0)$ in steady flow when $\beta'(t) = 0$.

Equation (38) is identical to that considered by Tomotika and Tamada[17] in their study of transonic nozzle flow, and has the solution

$$(Z - 4bS)^2 (Z + 2bS) = (\alpha^3 / 4b^3) \tag{39}$$

with α a constant of integration. The transverse velocity, $v^{(1)}$, determined from the irrotationality condition, $u_y^{(1)} = v_x^{(1)}$, is

$$v^{(1)} = 2byZ + 8b^2xy + (8b^3y^3/3) + y(8b^2\beta - 4k^2\beta'' + Q) \tag{40}$$

It is significant that while the reaction term $Q(t)$ and the function $\beta(t)$ are absent from Eqs. (38) and (39), these terms do appear in the expression for $v^{(1)}$. While Eqs. (39) and (40) describe the flow in a nozzle, the nozzle wall contour cannot be specified arbitrarily, but rather is determined from the similarity solution by combining Eq. (40) with the boundary condition [Eq. (24)] and integrating the resulting differential equation[13] for $F_w(x,t)$. In general the nozzle walls will not be stationary since the contour function will depend on both x and t.

When the integration constant $\alpha = 0$, it follows from Eq. (39) that

$$Z = 4bS \tag{41}$$

is a solution, and describes a flow accelerating from subsonic to supersonic velocity through a nozzle throat. Since

$$u_x^{(1)}(x,0,t) = 4b \tag{42}$$

in this case, the constant b here determines the velocity gradient on the nozzle axis. This simple, though physically significant, solution will now be considered in detail.

The transverse velocity $v^{(1)}$ now will be

$$v^{(1)} = 16b^2xy + \frac{32}{3}b^3y^3 + y(16b^2)\beta - 4k^2\beta'' + Q \tag{43}$$

If the arbitrary function $\beta(t)$ is chosen to make the coefficient of y in the last term equal to some constant C, both $v^{(1)}$ and the shape function F_w will be independent of time. The similarity solution now will describe flow through a nozzle with rigid walls, and this is the solution of greatest interest here. The function $\beta(t)$ must then satisfy the differential equation

$$4k^2\beta'' - 16b^2\beta = Q(t) + C \tag{44}$$

The contour of the rigid nozzle corresponding to this case has been determined by Adamson[13] in the absence of any heating, i.e., with $Q = 0$. The constant C establishes the origin of the x coordinate system and can be taken as zero for convenience.

The function $\beta(t)$ is now related to the heating function through Eq. (44) which then establishes the influence of $Q(t)$ upon the similarity solution. Equation (44) is a linear nonhomogeneous ordinary differential equation and so poses few difficulties. The result that the influence of heating enters through the boundary condition fits with the well-known analogy between heating and area change in one-dimensional compressible flow.[9]

Similarity Solution—Results

Since Eq. (44) for $\beta(t)$ is linear, it can be solved for any $Q(t)$ that can be expressed in terms of a Fourier series or integral. Hence it is first of interest to treat the case when $Q(t)$ is the harmonic function

$$Q(t)=Re(Ae^{i\omega t}) \tag{45}$$

where the amplitude $A\sim 0(1)$, and ω is a dimensionless angular frequency. The solution for $\beta(t)$ is then

$$\beta(t)=K_1e^{\frac{2b}{k}t}+K_2e^{-\frac{2b}{k}t}-Re\left[\frac{A}{4k^2\omega^2+16b^2}e^{i\omega t}\right] \tag{46}$$

Equation (36) for $u^{(1)}$ then yields the result

$$u^{(1)}=4bx+8b^2y^2+8bK_2e^{-\frac{2b}{k}t}$$

$$+Re\left[\frac{A}{4b\sqrt{1+\eta^2}}\exp[i(\pi+\omega t-\Psi)]\right]$$

$$\eta=(2k\omega)/4b;\quad \Psi=\tan^{-1}\eta \tag{47}$$

This solution for $u^{(1)}$ contains a transient part that decays exponentially. The value of the constant k defined by Eq. (11) has no effect on the final results and so will be taken as unity.

To assess the full meaning of this solution for $u^{(1)}$, it will be expressed in terms of the physical variables $\tilde{x}=(\bar{x}/\bar{\ell})$, $\tilde{y}=(\bar{y}/\bar{\ell})$, and $\tilde{t}=(\bar{t}/\bar{\tau}_f)$ which are related to the nozzle geometry and the critical speed of sound $\bar{a}^*$. Defining the dimensionless curvature $\kappa=\bar{\ell}/\bar{R}$, it readily is shown that

$$\kappa=16E_1b^2 \tag{48}$$

The constant b thus relates the small parameter E_1 to κ but does not affect the final result. In the present case, by letting $b=1/4$, Eq. (48) becomes $E_1=\kappa$. The relation between the dimensionless amplitude A and the actual heating rate is established by Eq. (34).

The dimensionless frequencies f [Eq. (27)] and ω become

$$f=\frac{\bar{f}\bar{\tau}_f}{\sqrt{2\kappa}}=\bar{f}\bar{\tau}_{ch}\sqrt{\frac{\gamma+1}{2}};\quad \omega=2\pi f\sqrt{\frac{2}{\gamma+1}} \tag{49}$$

In terms of physical variables, the solution now becomes

$$E_1u^{(1)}=\tilde{x}\sqrt{\frac{\kappa}{\gamma+1}}+\frac{\kappa}{2}\tilde{y}^2+2K_2\kappa\exp[-\tfrac{1}{2}\sqrt{\kappa(\gamma+1)}\,\tilde{t}]$$

$$+Re\left[\frac{\kappa A}{\sqrt{1+\eta^2}}\exp[i(2\pi f\sqrt{\kappa}\,\tilde{t}+\pi-\Psi)]\right]$$

$$\eta=4\pi f\sqrt{\frac{2}{\gamma+1}};\quad \Psi=\tan^{-1}\eta \tag{50}$$

The nozzle contour will be described by the equation[13]

$$\tilde{y}=1+\frac{\kappa}{2}\left[\tilde{x}+\frac{1}{6}\sqrt{(\gamma+1)\kappa}\right]^2 \tag{51}$$

from which it can be seen that the throat occurs at

$$\tilde{x}=-\left(\frac{1}{6}\right)\sqrt{(\gamma+1)\kappa}$$

Equation (50) describes the response of the flow near the throat of a converging-diverging nozzle to oscillatory, uniformly distributed, heating. This solution will be consistent with the ordering in the derivation provided $\tilde{x}\sim 0(\sqrt{\kappa})$, $\tilde{y}\sim 0(1)$, $f\sim 0(1)$ and $A\sim 0(1)$. The constant K_2 will depend on the initial conditions of the flow, but has no further effect after the decay of the transient term. From Eq. (50) it follows that $2\bar{\tau}_f/\sqrt{\kappa(\gamma+1)}=2\bar{\tau}_{ch}$ is the e folding time $\bar{\tau}_e$ of the transient. For air at standard conditions with $\bar{\ell}=5$ cm, and $\kappa=0.1$, this decay time is about 700 μs. Values of $\bar{\tau}_{ch}=(1/2)\bar{\tau}_e$ for the nozzles considered by Wegener and Cagliostro[5] and Barschdorff are shown in Table 1. It can be seen that the dimensionless frequency f, that determines the factor η, is a key parameter since it determines both the amplitude and the phase of the velocity oscillations induced by the heat input. Again, considering $\bar{\ell}=5$ cm, $\kappa=0.1$, the actual frequency $\bar{f}$ will be 2700 Hz when $f=1.0$.

The sonic line, where $Eu^{(1)}=0$, is a parabola that is concave in the upstream direction and oscillates about the nozzle throat. After the decay of the transient, the sonic line satisfies the equation

$$\tilde{x}\sqrt{\frac{\kappa}{\gamma+1}}+\frac{\kappa}{2}\tilde{y}^2=\frac{\kappa A}{\sqrt{1+\eta^2}}\exp i(2\pi f\tilde{t}\sqrt{\kappa}-\Psi) \tag{52}$$

Two additional examples of the heating function $Q(t)$ will be considered: a periodic square wave and a discontinuous change in Q. The square wave function

$$Q=A;\quad 0<\omega t<\pi,\quad Q=0;\quad \pi<\omega t<2\pi \tag{53}$$

can be represented by the Fourier series

$$Q=\frac{A}{2}+\frac{2A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t \tag{54}$$

The coefficients of the corresponding Fourier series for $\beta(t)$ can be found by substituting Eq. (54) in Eq. (44). With $\beta(t)$ known, $u^{(1)}(x,y,t)$ is readily determined and is now given by

$$u^{(1)}=4bx+8b^2y^2$$

$$-\left\{\frac{A}{2}+\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sqrt{1+\eta_n^2}}\cos[(2n-1)\omega t+\Psi_n]\right\}$$

$$\eta_n=(2n-1)\eta;\quad \Psi_n=\tan^{-1}\frac{1}{\eta_n} \tag{55}$$

The variation of $u^{(1)}$ at the origin $x=0, y=0$ is shown in Figs. 2a and b for the value of 1.0 and 2.0 for the frequency factor η. As is to be expected, the response of the flow to the oscillatory heat input decreases with increasing η and, hence, frequency f. For $\eta\ll 1$, it is clear from both the simple harmonic and square wave solutions that the oscillations of $u^{(1)}$ will have the same form as those of the imposed heat input $Q(t)$. As an example for $\eta=0.1$, $\bar{a}^*\simeq 300$ m/s, $\bar{\ell}=5$ cm, it turns out that $f=8.7\times 10^{-3}$ and the actual frequency $\bar{f}$ will be 23 Hz.

In the case of a discontinuous change in Q such that

$$Q=0;\quad t<0,\quad Q=A;\quad t>0 \tag{56}$$

The solution for $u^{(1)}$ for $t>0$ is given by

$$u^{(1)}=4bx+8b^2y^2+\frac{A}{4b}(e^{-2bt}-1) \tag{57}$$

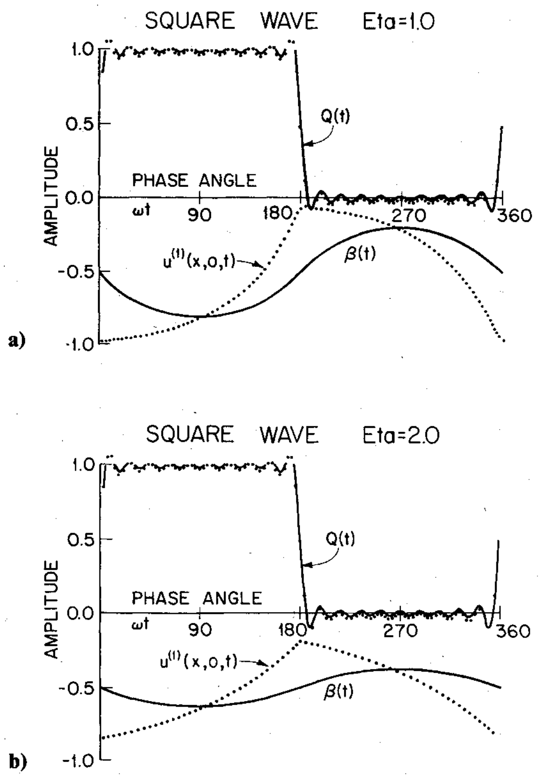

Fig. 2 Square wave response; a) $\eta = 1.0$ and b) $\eta = 2.0$.

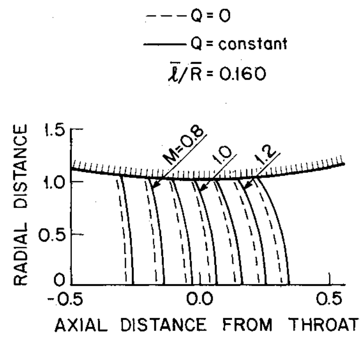

Fig. 3 Lines of constant M and Q=const.[7]

Here again the e folding time for the flow to adjust to the sudden change in Q is $2\bar{\tau}_{ch}$.

Generally, shock waves which may move through the throat region will occur in unsteady nozzle flow, and are present in the interferograms of flow with condensation[2] mentioned formerly. Such shock waves can be incorporated in the similarity solution previously described.[18,19] The difficulty lies in developing solutions in which the nozzle walls are stationary both upstream and downstream of the shock.

Adamson's results[18] for adiabatic flow can be used to discuss the present case. Even with shocks the solution is still described by Eqs. (36-38) with $Z(S)$ given by Eq. (39). However, $Z(S)$ is discontinuous at the shock which lies on a parabola $S = S_0 = \text{const}$. The Hugoniot conditions across the shock require that

$$Z_d(S_0) = -Z_u(S_0) \tag{58}$$

where subscripts u and d refer to conditions upstream and downstream of the shock. Taking the upstream solution as the accelerating flow $Z_u = 4bS$, already considered, the downstream solution is given by Eq. (39) with the integration constant, $\alpha = \alpha_d < 0$. The shock condition then requires that

$$S_0 = -(\alpha_d / 8b^2) \tag{59}$$

Upstream of the shock the solution will be identical to that treated formerly with $\beta_u(t)$ and $Q(t)$ related by Eq. (44). The fact that $S = S_0 = \text{const}$ on the shock implies that the shock moves in the x direction with velocity $\bar{u}_s$ given by

$$\bar{u}_s = -kE_1(\gamma+1)\bar{a}^*\beta_u'(t) \tag{60}$$

The shock is thus "driven" by the heat input $Q(t)$ through the function $\beta_u(t)$. The problem is that since $Z(S)$ changes discontinuously across the shock, the transverse velocity $v^{(1)}$ given by Eq. (40) also changes and becomes time dependent. Thus, with $\beta_u(t)$ determined from $Q(t)$ in the flow upstream of the shock, the nozzle contour downstream will vary with time. A similarity solution with shock waves and rigid nozzles walls, thus, does not appear feasible.

Discussion

Equations have been formulated for the unsteady slowly time varying transonic flow of a gas with external or reactive heating. These equations are identical to the nonreactive or adiabatic unsteady transonic equations except for the inclusion of a heating term. Consideration of experimental results suggests that these equations are appropriate for the analysis of the condensation induced oscillations observed in transonic nozzles. However, the treatment of such flows will require a consideration of shock waves and condensation fronts in conjunction with the equations derived here.

It has been possible to develop a similarity solution for a uniform, but time varying, heat input as might arise in electrical or Joule heating. It is interesting that the response of the flow to heating is established through a linear equation that arises from the boundary conditions at the walls of the nozzle. The unsteady flows due to both a periodic and stepwise heat input have been determined. Transient disturbances are found to decay in a time of the order $\bar{\tau}_{ch} = \bar{\ell}/\bar{a}^*\sqrt{\kappa(\gamma+1)}$. For a periodic input with frequency $\bar{f}$, the parameter $\eta = 4\pi\bar{f}\bar{\tau}_{ch}$ determines the response of the flow. For a large frequency in the sense $\eta \gg 1$, the response of the flowfield will be minimal.

There appear to be no experimental results corresponding to the self-similar solution described here. However, one of Novak's[7] finite-difference calculations (or computer experiment) of nozzle flow with heating corresponds to a discontinuous change of Q from zero to a small finite value. The computations were for an axisymmetric nozzle making direct comparison with the present analysis impossible. However, Novak's results are in qualitative agreement with the results presented here. Figure 3, which is taken from Novak[7] shows lines of constant Mach number near the nozzle throat for $Q = 0$ and for a constant Q after equilibrium has been established. The nozzle in question has $\bar{\ell} = 0.01$ m and $\bar{\ell}/\bar{R} = 0.160$. The constant heating rate causes the constant Mach number lines to shift downstream without change of shape, and this behavior is precisely that indicated by Eq. (57) in the limit $t \to \infty$. Novak's results required extensive numerical computations while the analysis here, although limited in applicability, is extremely simple.

Similarity solutions with shock waves are readily determined; however, it has not been possible to develop such solutions with nozzle contours that are rigid throughout. To deal with such flows, a technique similar to that described by Adamson et al.[19] is probably appropriate.

The viscous boundary layer at the nozzle wall significantly can affect the flow, especially when shock waves are present. However, the theory presented here is essentially an inviscid one so that the equations cannot satisfy the no-slip condition. The core flow-boundary layer interaction is beyond the scope of the present paper.

In this paper, particular emphasis has been placed on nozzle flows with condensation or external heating. However, the analysis will also be applicable in the case of exothermic chemical reactions if ξ denotes a reaction progress variable and $\bar{L}$ is replaced by the enthalpy of reaction. The physics of the process will, of course, be quite different.

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